1. (10pts) Let $\Sigma = \{a, b\}$ be an alphabet and let $f : \Sigma^* \to \mathbb{N}$ be a function from $\Sigma^*$ to nonnegative integer which counts the number of $a$s in the input. $f$ can be defined inductively as follows:

Basis case: If $x = \varepsilon$ then $f(x) = 0$.

Inductive case: 1. If $x = ay$, then $f(x) = 1 + f(y)$, and
2. If $x = by$, then $f(x) = f(y)$.

Now prove by structural induction (on $x$) that for all strings $x, y$, $f(xy) = f(x) + f(y)$.

2. (6 pts) Let $A$ and $B$ are two sets of strings. What are the sets $A \cup B$ and $A^*$?

(a) $A \cup B = \text{def } \ldots$ \\
(b) $A^* = \text{def } \ldots$

3. (10 pts) Let $A = \{aa, aaaa\}$ and $B = \{aaa, aaaa\}$. Then which of the following statements are correct?

(a) $a^{10} \in A^*$.
(b) $a^{10} \in AB$.
(c) $\varepsilon \in A^*$.
(d) $A^* = AA^* \cup \{\varepsilon\}$.
(e) $(B^*)^* = B^*$

ANS: ____________________________

4. (39 pts) Given an NFA $M$ defined as follows:

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>1F</td>
<td>2</td>
<td>[2]</td>
<td>[1]</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>[3]</td>
<td>{}</td>
</tr>
<tr>
<td>3F</td>
<td>{}</td>
<td>{}</td>
<td>[2,3]</td>
</tr>
</tbody>
</table>
(a) Draw a state transition diagram for the NFA M. [8 pts]

(b) Let M = (Q, Σ, δ, S, F). Then what are the contents of Q, Σ, δ, S and F, respectively. [7 pts]

Q = ______________________________
Σ = _____________________________
S = ______________________________
F = ______________________________
δ =

(c) List 4 strings of different length accepted by the machine. [4 pts]

ANS: ______________________________________________________

(d) Construct a DFA equivalent to the above NFA. Remember to indicate the set of old states each state of the new machine corresponds to. [8 pts]

(e) Find a regular expression equivalent to the FA. [7 pts]
(f) Write a strongly left linear grammar equivalent to the FA. [5pts]

5. [10 pts] Let $A = \{ a^{nx} | n > 0 \}$ be the set of strings of as of length $n^2$. Use (game-theoretic) pumping lemma to show that $A$ is not regular.

6. [6 pts] Consider the grammar $G$:

$$
S \rightarrow ST \mid a \quad T \rightarrow bS
$$

Show that the input ‘ababa’ is a member of $L(G)$ by giving a derivation of ‘ababa’ from $S$. Note that the derivation must be complete and is not shortcut.

7. (14 pts) CFG Design:

(a) Design a CFG to generate the set $\{ a^n b^n | n \geq 0 \}$

(b) Design a CFG to generate the set $\{ a^m b^n | m, n \geq 0 \text{ and } m \neq n \}$
8. (10 pts) Consider the following pair of machines:

\[
\begin{array}{c|cc}
M1 & a & b \\
\hline
>1 & 2 & 3 \\
2 & 3 & 1 \\
3F & 2 & 3 \\
\end{array}
\]

\[
\begin{array}{c|cc}
M2 & a & b \\
\hline
>1 & 2 & 1 \\
2F & 1 & 2 \\
\end{array}
\]

Use the product construction to construct a DFA that accepts the set \( L(M_1) - L(M_2) \). Remember to mark initial and final states of the new machine.

9. (15 pts) Consider the context free grammar \( G \):

\[
S \rightarrow [S] \mid SS \mid \varepsilon
\]

(a) Devise a grammar \( G_1 \) such that \( L(G_1) = L(G) - \{\varepsilon\} \) and \( G_1 \) contains no empty rule.[7pts ]

(b) Design a grammar in Chomsky normal form which is equivalent to \( G_1 \).